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A TWO-STEP BILINEAR FILTERING APPROXIMATION

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ABSTRACT

A new approximation technique to a certain class of nonlinear filtering (signal processing) problems is considered here. The method is based on an approximation of a nonlinear, partially observable system by a bilinear model with fully observable states. The filter development proceeds from the assumption that the unobservable states are conditionally Gaussian with respect to the observation initially. The method is shown to be promising for real-time communication and sonar applications as demonstrated by computer simulations. Moreover, some of the traditional techniques evolve as special cases of this methodology.

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I. INTRODUCTION

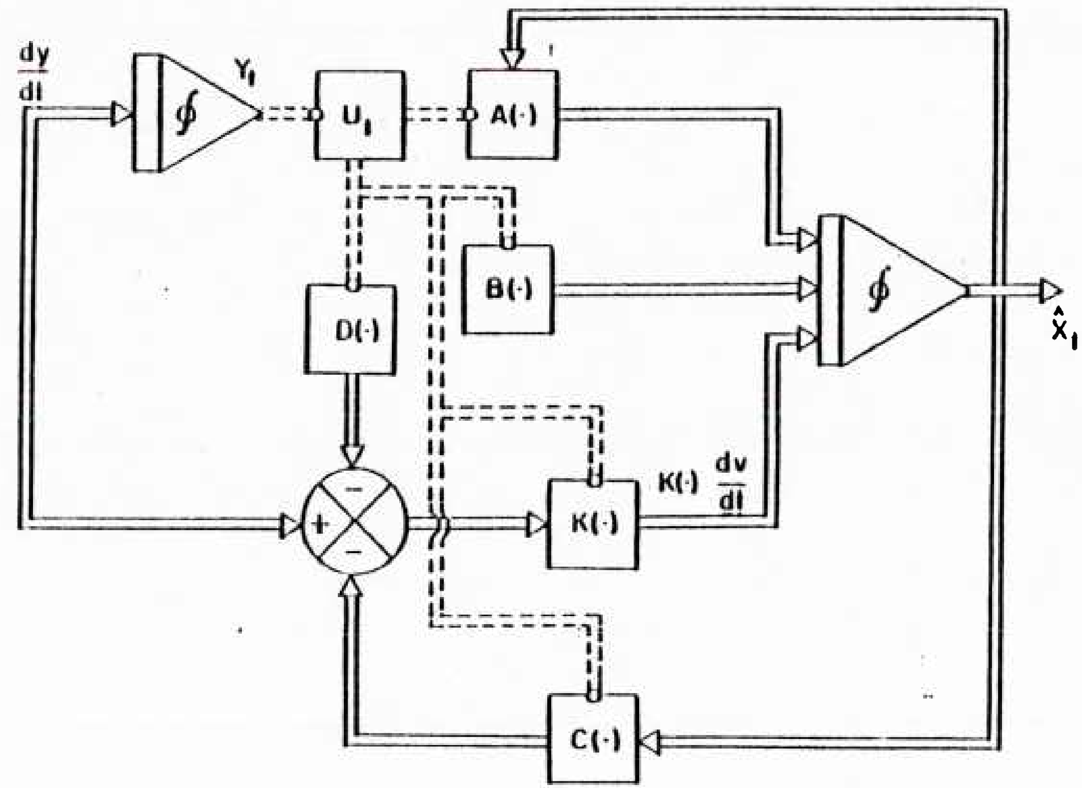
In general, the optimal nonlinear mean square filtering (signal processing) problem does not have a finite-dimensional recursive synthesis. This is due to the fact that the conditional probability density of the unobservable states with respect to the observations cannot, in general, be characterized by a finite parameter set, such as for a Gaussian distribution utilizing linear methods. Thus, approximation and ad-hoc techniques must be employed to construct practical filters for nonlinear systems.

In this paper a new nonlinear finite-dimensional filtering methodology is presented. The method which could be called a two-step nonlinear filter (TNF) is of a more general nature than the linear model approach (or extended linear filtering), and does not require the model "smoothness" which is crucial to most of the existing techniques. Hence, the proposed technique may expand the range of practical problems that can be handled by nonlinear filtering.

Herein, the general nonlinear model is approximated by a control model of "bilinear form." The "best" model approximation of this form is then computed by the appropriate stochastic control [1]. The final step, with the feedback control as a function of the estimated state $\hat{x}_t$, requires computation of the optimal (m.s.e.) state estimator by the conditionally Gaussian filter which is formally developed by Liptser and Shiryayev [2] and extended by Kolodziej [3].

Extension of the results to large-scale nonlinear systems is accomplished by incorporating a novel decomposition scheme in the filter design to alleviate the complexity of the control problem and the "curse of dimensionality" of the filtering algorithm. Formal representation of the filter is given in Figure 1.

Application of the developed filter to a scalar nonlinear system which lacks model smoothness is discussed and application of the derived multi-dimensional filtering algorithm to a low-order nonlinear tracking problem according to a global criterion is presented. In addition, a comparison with traditional methods, such as the popular Extended Kalman Filter (EKF) is given by digital computer simulation to demonstrate the effectiveness of the obtained results.



NOTE: The double lines are for vector representation. The symbol $\oint$ refers to stochastic integration. $\Longrightarrow$ *Vector multiplication.*

The dashed lines implies functional representation.

Figure 1. Schematic Diagram for the Proposed Conditionally Gaussian Filter.

II. MODEL APPROXIMATION AND FILTER STRUCTURE

Consider, as given, some complete probability space (Ω,F,P). Let (w_t^i), $i = 1,2$ defined on (Ω,F,P) be mutually independent Wiener processes. Also, given the following nonlinear models, which may represent a broad class of nonlinear systems, with dynamics described by a family of stochastic differential equations (in the Ito sense) of the form

$$
\begin{aligned}
dx_t &= F(x,y,t)dt + G(x,y,t)dw_t^1 , \\
dy_t &= H(x,y,t)dt + R(x,y,t)dw_t^2 ,
\end{aligned}
\tag{1}
$$

where $(F(\cdot), H(\cdot))$ are nonlinear, real vector functions, and $(G(\cdot), R(\cdot))$ are matrices of compatible orders.

The optimal m.s.e. filter for the system in (1) is known to be the conditional expectation of the system given the observation $(y_s;\ 0 \leq s \leq t)$, $t \in [0,T]$ provided that a finite second moment solution exists with

$$
\hat{x}_t = E(x_t/y_s;\ s \in [0,t]) . \tag{2}
$$

In principle, a sequential version of (2) can be found, but in general, the recursive formulae consist of an infinite-dimensional system of moment equations which are needed to characterize completely the conditional probability density, $p(x_t,t/y_s;\ s \in [0,t])$. Thus, one is forced to seek an approximation technique for practical implementation.

Herein, a new technique, which generates a finite-dimensional, nonlinear filter and a "close" (m.s.e.) model approximation to the original model in (1) is presented. This technique may be called "an approximation in the model parameter space," and its parameters are functions of the feedback control law, u_t, which is itself a function of the observation process y_t.

In general, the model in (1) can be approximated as follows:

$$dx_t \simeq \tilde{F}(x_t,u_t,t)dt + \tilde{G}(u_t,t)dw_t^1 ,$$
$$dy_t \simeq \tilde{H}(x_t,u_t,t)dt + \tilde{R}(u_t,t)dw_t^2 , \tag{3}$$

where $(\tilde{F},\tilde{G},\tilde{H},\tilde{R})$ are of appropriate dimensions and are functionals of u_t. The control u_t is a measurable stochastic process with respect to Y_t = σ-algebra$(y_s$; $s \in [0,t]$, and it is chosen to minimize the following mean-square (global) criterion:

$$Q(u) = E\left(\int_0^T \| k_t - \tilde{k}_t \|^2 \, dt \right) , \tag{4}$$

where k_t denotes any of the functions F, H, R, or G in (1), and $\tilde{k}_t$ is its corresponding approximation in (3). Here $\| \cdot \|$ is the Euclidean norm, and the arguments (x_t,y_t,t), (x_t,u_t,t) are omitted for brevity. It is noteworthy that the choice of (4) ensures a form of global filtering criterion even though the search for the approximation $\tilde{k}_t$ falls into a class of stochastic control problems and depends strongly on the type of nonlinearities in the system.

An important special class of (3) is the following bilinear form:

$$dx_t \simeq [A(u_t,t)x_t + B(u_t,t)]dt + \tilde{G}(u_t,t)dw_t^1 ,$$

$$dy_t \simeq [C(u_t,t)x_t + D(u_t,t)]dt + \tilde{R}(u_t,t)dw_t^2 , \tag{5}$$

where $(A,B,C,D,\tilde{G},\tilde{R})$ are linear functionals of u_t and are of appropriate dimensions. The term bilinear refers to the fact that the system is linear in control and state, but not jointly linear [4]. It has been shown that such bilinear systems are quite common in physical applications and represent a significant class in their own right.

Now assume $\hat{u}_t$, which minimizes (4), is available; then (5) is a "close," in the sense of (u), approximation model to system (1) since the minimization criterion is a measure of the quality of the approximation. If it is assumed that the initial unobservable state, x_0, is conditionally Gaussian with respect to the initial observable state, y_0, then under certain broad assumptions (see [2] or [3]) the optimal m.s.e. filter for the system in (5) takes the form of

$$\begin{aligned}
d\hat{x}_t &= (A\hat{x}_t + B)dt + \Lambda dv_t , \\
d\Gamma_t &= (A\Gamma_t + \Gamma_t A^* + \tilde{G}\,\tilde{G}^* - \Lambda\Lambda^*)dt , \\
\Lambda &= (\Gamma_t C^*)\bar{R}^{-1} , \\
dv_t &= \bar{R}^{-1}[dy_t - (C\hat{x}_t + D)dt] , \\
\bar{R}^2 &= \tilde{R}\,\tilde{R}^* ,
\end{aligned} \tag{6}$$

where $\hat{x} = E(x_t/Y_t)$, $\Gamma_t = \mathrm{Cov}\ (x_t/Y_t)$.

Here $E(/)$ denotes the conditional expectation operator, Γ_t conditional covariance, * transposition operator, and Y_t is the σ-algebra generated by the observations on $[0,t]$, $t \in [0,T]$. Again the arguments $(\hat{u}_t,t)$ are omitted for brevity.

Now, if the optimal control $\hat{u}_t$ or its approximation can be obtained, then system (6) provides a finite-dimensional approximation for the nonlinear filtering of (1). Also, the partially observable system in (5) is transformed into a completely observable system (6). Consequently, to solve for the control law analytically, the minimization criterion (4) must be transformed accordingly. Thus, if (4) is rewritten in the form

$$Q(u) = E \int_0^T L(x_t,u_t,t)dt \ ,$$

then for $u = \hat{u}$

$$Q(u) = E(\int_0^T \hat{L}(\hat{x}_t,\Gamma_t,u_t,t)dt) \ , \tag{7}$$

where $\hat{L}(\hat{x}_t,\Gamma_t,u_t,t) = \int_{-\infty}^{\infty} L(\xi,u_t,t)f(\xi,\hat{x},\Gamma_t)d\xi$, and $f(\xi,\hat{x},\Gamma_t)$ denotes a differential Gaussian measure with mean $\hat{x}_t$, variance Γ_t.

If $\hat{u}$ is replaced in (6) by u, then an equivalent, completely observable, stochastic control problem [(6), (7)] emerges where the new state of the system $(\hat{x}_t,\Gamma_t)$ are generated by (6). Hence, the filtering problem is actually replaced by a stochastic control problem which results in a stochastic nonlinear Bellman equation [1]. Apparently, this could lead to a more difficult problem to solve than the original filtering problem for (1). However, satisfactory approximation to the control law can be found without solving the Bellman equation exactly.

It might be noted here that the solution to this problem according to $\hat{u}_t$ resembles the approximation criterion in the EKF approach, and makes the approximation of closed analytical form.

The following scalar example illustrates the suggested filtering procedures.

Example 1

Consider a process with an absolute value detector so that

$$\begin{aligned} dx_t &= f|x_t|dt + \sigma_1\, dw_t^1 , \\ dy_t &= hx_t\, db + \sigma_2\, dw_t^2 , \end{aligned} \tag{8}$$

where w_t^1, w_t^2 are independent Wiener processes, f,h,σ_1,σ_2 are constant, and $t \in [0,T]$. The bilinear feedback approximation yields

$$\begin{aligned} dx_t &\simeq \hat{u}_t\, x_t dt + \sigma_1\, dw_t^1 , \\ dy_t &\simeq x_t dt + \sigma_2\, dw_t^2 , \end{aligned} \tag{9}$$

where $\hat{u}$ is selected to minimize

$$E\left(\int_0^T (f|x_t| - \hat{u}x_t)^2 dt\right) . \tag{10}$$

The optimal (m.s.e.) filter for (8) with x_0 conditionally Gaussian w.r.t., y_0 has the form of

$$d\hat{x}_t = \hat{u}x_1\ dt + \Gamma_t\ h(\sigma_2)^{-1}\ dv_t\ ,$$
$$d\Gamma_t = [2\hat{u}_t\Gamma_t + \sigma_1^2 - h^2\ \Gamma_t^2(\sigma_2)^{-2}]dt\ , \tag{11}$$

where $dv_t = (\sigma_2^{-1})(dy_t - h\hat{x}_t)dt$ is the innovation process. Now from (7) the equivalent minimization criterion to (10) is

$$\bar{Q}(t,\hat{x},t) = E(\int_0^T (u|\xi| - \hat{u}\xi)^2 d\Phi(\xi)\ , \tag{12}$$

where $d\Phi(\xi) = \frac{1}{\sqrt{2\pi\Gamma_t}}\exp(-0.5\frac{(\hat{x}-\xi)^2}{2\Gamma_t})d\xi$, for nonsingular Γ_t. From [5] it is found that the optimal $\hat{u}$ is given by

$$\hat{u} \simeq f[\frac{2\hat{x}\Gamma_t}{\sqrt{2\pi\Gamma_t}}\exp(-\frac{\hat{x}^2}{2\Gamma}) - 2(\hat{x}^2 + \Gamma)\mathrm{erf}(-\frac{\hat{x}}{\sqrt{2\Gamma}}) - (\Gamma\frac{\partial v}{\partial\Gamma} + 0.5\hat{x}\frac{\partial v}{\partial\hat{x}}](\hat{x}^2 + \Gamma)^{-1}\ , \tag{13}$$

where v is the solution of the stochastic Bellman equation which means that we have to solve a nonlinear partial differential equation first in order to evaluate the optimal control in (13). However, it is seen that an approximating optimal control

$$\tilde{u} = f\ \frac{[2\hat{x}\Gamma\exp(-\hat{x}/2\Gamma)]}{\sqrt{2\pi\Gamma}\ (\hat{x}^2+\Gamma)} + 2\ \mathrm{erf}(-\hat{x}/\sqrt{2\Gamma}) \tag{14}$$

already gives better performance than the EKF [5].

III. DECOMPOSITION SCHEME FOR MULTIDIMENSIONAL SYSTEM

Extension of the previous results to large-scale nonlinear systems is accomplished by incorporating a novel decomposition scheme in the filter design. This should alleviate the "curse of dimensionality" which is encountered if the optimal approximation is applied directly. Recall that this solution involves a function of $(n + \frac{n(n+1)}{2})$ variables, where n is the dimension of the system.

The strategy adopted in this decomposition scheme is based on the decomposition of the original system into two interconnected subsystem approximations. The first subsystem is only linear (i.e., linear process and observation equations). The second subsystem includes all the nonlinearities in the system which are then approximated by the proposed TNF. The main advantage of using this scheme is that the control parameters, which are needed in the TNF algorithm, will be easily obtained as functions of the parameters of the first-stage linear filter, S_0. Figure 2 shows a block diagram representation of the suggested scheme.

An interesting broad class of nonlinear systems of the following form is considered here.

$$dx_t = F(x,y,t)dt + G(x,y,t)dw_t \,,$$
$$dy_t = H(x,y,t)dt + \sigma(t)dv_t \,, \tag{15}$$

where w_t, v_t are mutually independent vector Wiener processes of appropriate

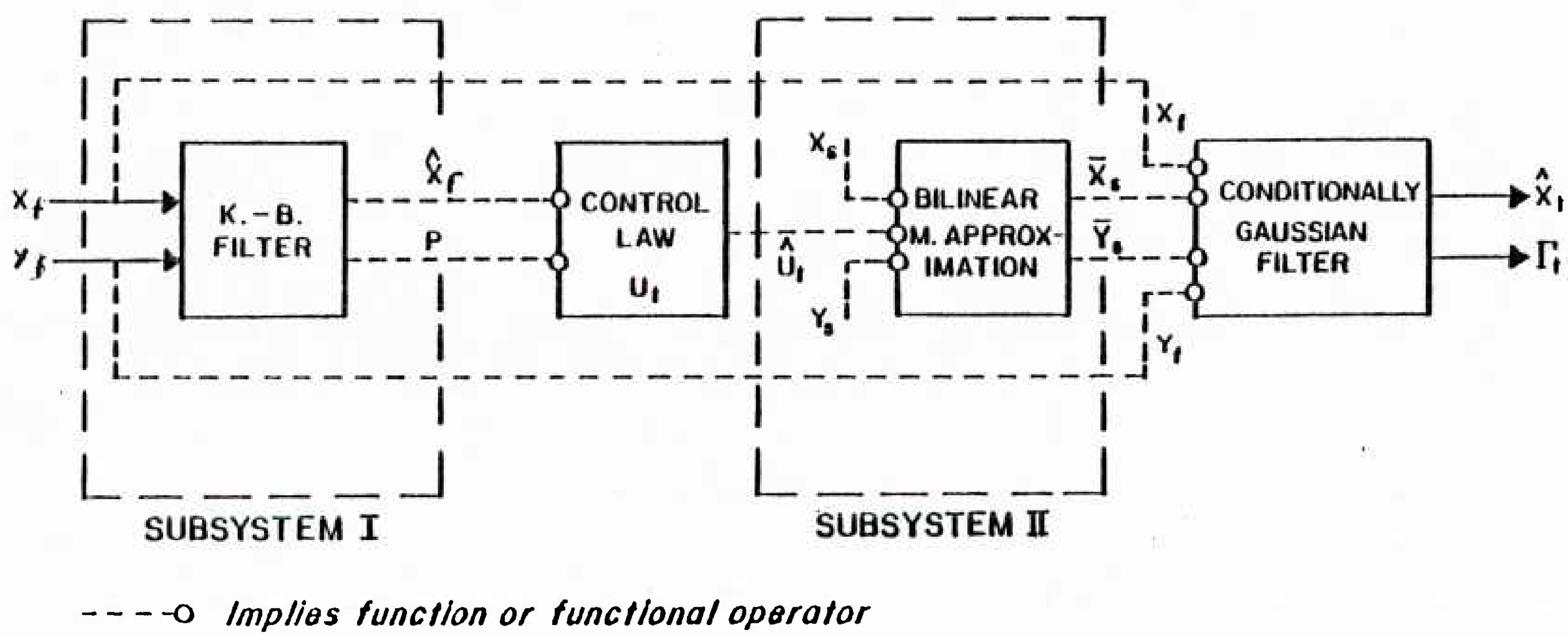


Figure 2. Block Diagram Representation of the TNF.

dimensions, $\sigma(t)$ is a matrix of compatible order. The functions $F(\cdot)$, $H(\cdot)$, and $G(\cdot)$ can be partitioned as follows:

$$F(x,y,t) = f_1(t)x + f_2(x,y,t) \ ,$$

$$G(x,y,t) = g_1(t) + g_2(x,y,t) \ , \tag{16}$$

$$H(x,y,t) = \begin{bmatrix} h_1(t)x \\ h_2(x,y,t) \end{bmatrix} , \ \sigma(t) = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} .$$

Here f_1, h_1, and g_1 are matrices of appropriate dimensions. f_2, h_2, and g_2 are nonlinear functions of their arguments, and of compatible orders.

The general outline of the decomposition scheme and the various filtering algorithmic steps are:

(1) The nonlinear system as in (15) can be decomposed into two subsystems.

<u>Subsystem I</u>

$$dx_{ft} = f_1(t)x_{ft} + g_1(t)dw_t \ ,$$
$$dy_{ft} = h_1(t)x_{ft} + \sigma_1(t)dv_t^1 \ , \tag{17}$$

where f_1, h_1, and g_1 are as defined before, and w_t, v_t^1 are independent Wiener processes of appropriate dimensions. The subscript f denotes the first subsystem.

Subsystem II

$$dx_{st} = f_2(x,y,t)dt + g_2(x,y,t)dw_t \ ,$$

$$dy_{st} = h_2(x,y,t)dt + \sigma_2 \, dv_t^2 \ , \tag{18}$$

where f_2, g_2, and h_2 are as defined before and w_t, v_t^2 are again independent Wiener processes of compatible orders. The subscript s denotes the second subsystem.

(2) Apply a classical filtering technique, i.e., the Kalman-Bucy algorithm [6] to the linear system in (17) as follows:

$$d\hat{x}_{ft} = f_1\hat{x}_f dt + ph^*(\sigma_1\sigma_1^*)^{-1} \, d\eta_t \ ,$$

$$dp_t = (f_1p + pf_1^* + g_1g_1^* - ph_1^*(\sigma_1\sigma_1^*)^{-1}h_1p)dt \ ,$$

$$d\eta_t = dy_{ft} - h_1\hat{x}_{ft}dt \ , \tag{19}$$

$$\hat{x}_f(0) = E(x_f(0)), \ p\,(0) = \mathrm{cov}(x_f(0)) \ ,$$

where $\hat{x}_f$ is the estimate, and p(t) is the error covariance matrix. This will be considered as the first stage of the filtering algorithm.

(3) Find an appropriate bilinear approximation model for the system in (18) of the following form:

$$dx_s \simeq \tilde{f}_2(x,u_t,t)dt + \tilde{g}_2(u_t,t)dw ,$$
$$dy_s \simeq \tilde{h}_2(x,u_t,t)dt + \sigma_2 dv_2 , \tag{20}$$

where

$$\tilde{f}_2(x,u_t,t) = \sum_{i=1}^{n} u_{ti}x_i + u_{t(n+1)} = A(u_t,t)x + B(u_t,t) ,$$

$$\tilde{h}_2(x,u_t,t) = \sum_{j=1}^{n} u_{tj}x_j + u_{t(n+2)} = C(u_t,t)x + D(u_t,t) ,$$

$$\tilde{g}_2(u_t,t) = u_{t(n+3)} = g_o(u_t,t) . \tag{21}$$

Here the second equality is used for mathematical convenience. The controls $u = \{u_j\}$, $j=1,2,\ldots n+3$ are measurable with respect to σ-algebra $\{y_{fs};\ s \in [0,t]\}$, and are chosen to minimize the following global filtering criterion

$$Q(u) = \min_u E\,[\int_0^T (k-\tilde{k})^2\,dt] , \tag{22}$$

where k denotes any of the functions f_2, g_2, or h_2, while $\tilde{k}$ denotes the corresponding approximation $\tilde{f}_2$, $\tilde{g}_2$, or $\tilde{h}_2$ in (21). Using the property of expectation and Bayes formula, (22) becomes

$$Q(u) = \min_u E(\int_0^T E(k-\tilde{k})^2/y_{fs};\ s \leq t)dt$$
$$= \min E(\int_0^T L(y_f,u)dt) , \tag{23}$$

where $L(y_f,u) = E^t(k-\tilde{k})^2$, and arguments (x,y,t), (u_t,t) are omitted for brevity.

The minimization of (23) with respect to $u = \{u_j\}$, $j=1,2,\dots n+3$ can be performed "locally" since L depends on the σ-algebra $(y_{fs}$, $s \in [0,t])$, which is not affected by the u_j's.
Assume that the $\hat{u}_j$'s which minimize (23) are obtained.

(4) The new equivalent system has the following form:

$$dx_t = (A_1(\hat{u}_t,t)x_t + B_1(\hat{u}_t,t)dt + G_1(\hat{u}_t,t)dw \,,$$
$$dy_t = (C_1(\hat{u}_t,t)x_t + D_1(\hat{u}_t,t))dt + \sigma dv \,, \tag{24}$$

where

$$A_1(\hat{u}_t,t) = [f_1(t) + A(\hat{u}_t,t)],\ B_1(\hat{u}_t,t) = B(\hat{u}_t,t) \,,$$

$$C_1(\hat{u}_t,t) = \begin{matrix} h_1(t) \\ C(\hat{u}_t,t) \end{matrix} \,,\ D_1(\hat{u}_t,t) = \begin{matrix} 0 \\ D(\hat{u}_t,t) \end{matrix} \,,$$

$$G_1(\hat{u}_t,t) = (g_1(t) + g_o(\hat{u}_t,t)),\ \sigma = \begin{matrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{matrix} \,.$$

Here again the matrices are of compatible orders.

(5) Again, with certain assumptions about $(A_1, B_1, C_1, D_1, G_1, \sigma)$ and the distribution of the initial state x_0 given y_0 (see [2] and [3]), the corresponding conditionally Gaussian filter is of the following form:

$$d\hat{x}_t = (A_1\hat{x}_t + B_1)dt + S\,dv\ ,$$

$$S = (\Gamma_t C_1^*)(\sigma\sigma^*)^{-0.5}\ ,$$

$$dv = (\sigma\sigma^*)^{-0.5}(dy_t - (C_1\hat{x}_t + D_1)dt)\ , \tag{25}$$

$$d\Gamma_t = (A_1\Gamma_t + \Gamma_t A_1^* + G_1G_1^* - SS^*)dt\ ,$$

where A_1, B_1, C_1, D_1 are as in (24), and the arguments $(\hat{u}_t, t)$ are again omitted for brevity. A schematic diagram representation of the algorithm is given in Figure 3. A second-order example to demonstrate the algorithm steps follows.

Example 2

A second-order, linear sonar target track is considered here with a non-linear observation. This could represent active tracking of a multi-mode range system or passive tracking with multi-receiver-transmitter and correlated time delay.

It is assumed that state vector (range = x_1, range rate = x_2) evolves according to the following stochastic differential equations:

$$dx = Fxdt + Gdw_t^1\ , \tag{26}$$

where $dx = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$, $\alpha = 1/\tau$.

w_t^1 is a Wiener process, and τ is the target maneuver time constant.

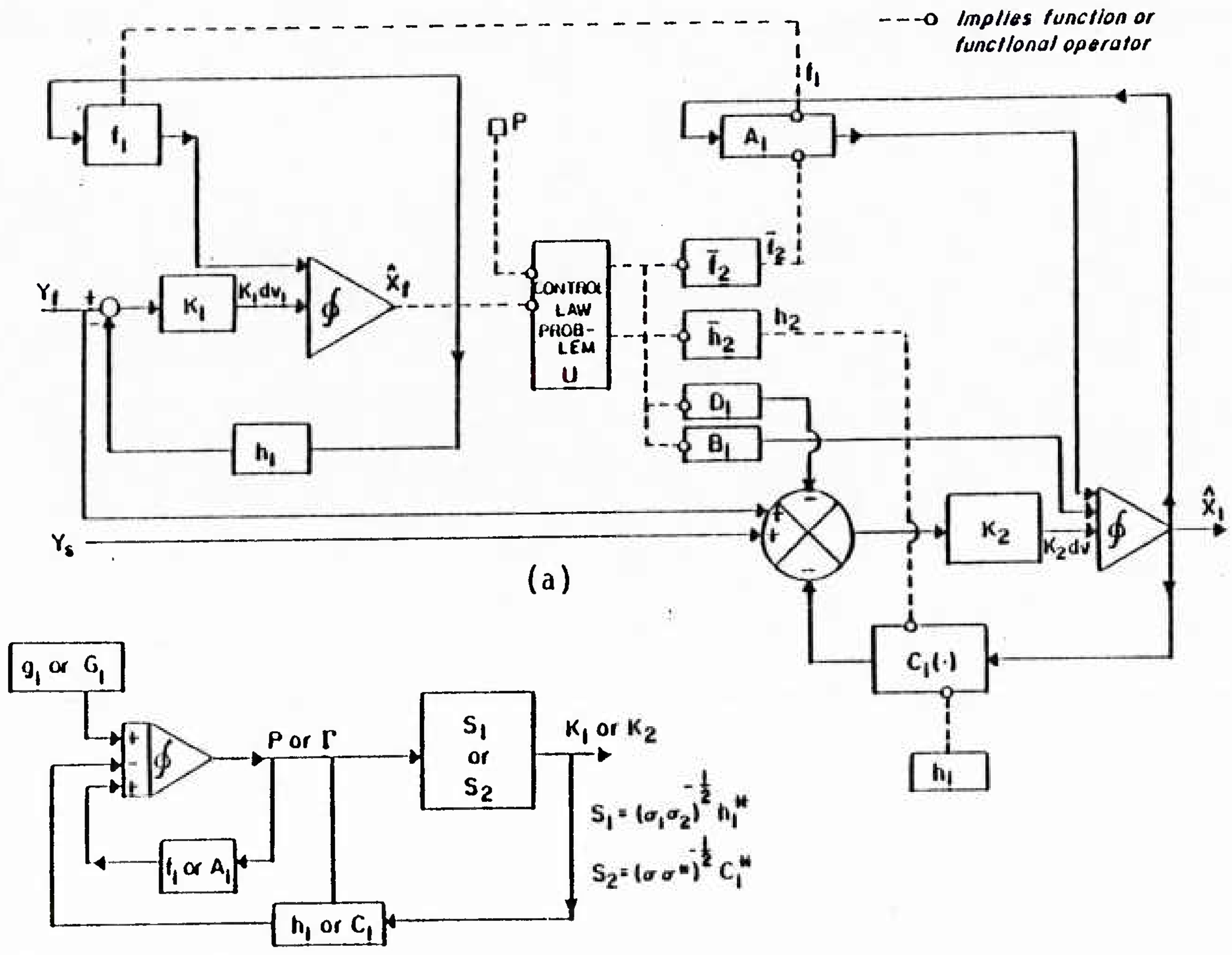


Figure 3. Schematic Representation for the TNF.

The measurement equations are nonlinear due to target motion during transmission. Neglecting the nonlinearities in the "velocity observation," it is assumed that

$$dy = H(x,t)dt + Rdw_t^2 , \tag{27}$$

where $dy = \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix}$, $H(x,t) = \begin{bmatrix} x_1 + mx_1x_2 \\ x_2 \end{bmatrix}$, $R = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, w_t^2 is a Wiener vector of measurement noises, $m = a/c$, a is a constant, and c is the average speed of sound in water. Applying the above algorithm, the two subsystems are given next.

<u>Subsystem I</u>

Here only the linear, y_2, observation is used. The system and observation equations are

$$\begin{aligned} dx_1 &= x_2\, dt , \\ dx_2 &= -\alpha x_2 + \alpha dw_t^1 , \\ dy_2 &= x_2 dt + \sigma_2 dw_t^2 . \end{aligned} \tag{28}$$

Then using the Kalman-Bucy filter for (28) yields

$$\begin{aligned} d\hat{x}_{1f} &= \hat{x}_{2f}dt + \frac{p_3}{\sigma_2}\, dv , \\ d\hat{x}_{2f} &= -\alpha\hat{x}_{2f}dt + \frac{p_2}{\sigma_2}\, dv , \end{aligned} \tag{29}$$

where $\hat{x}_{if} = E(x_i/y_2)$, $i = 1,2$, the conditional expectation, dv is the innovation process, and p_2, p_3 are the standard time-variant solutions to components of the Riccati equation.

Subsystem II

The best bilinear observation approximation is obtained such that u_j, $j = 1,2,3$ are measurable with respect to the σ-algebra (y_{2s}, $s \in [0,t]$), and minimize the following global criterion:

$$J(u) = \min_{u_j} E\left(\int_0^T (x_1x_2 - (u_1x_1 + u_2x_2 + u_3)^2)dt\right) , \quad (30)$$

which can be written equivalently using the properties of expectation as

$$J(u) = \min_{u} E\left(\int_0^T L(y,u)dt\right) , \quad (31)$$

where $L(y,u) = E^t(x_1^2x_2^2) - E^t(R_1) + E^t(N^2)$, $N = u_1x_1 + u_2x_2 + u_3$, $R_1 = 2x_1x_2N$. E^t refers to conditional expectation with respect to the observation Y_t. Thus, performing the minimization with respect to u_j, $j = 1,2,3$, the following are obtained (see [5]):

$$\begin{aligned} \hat{u}_1 &= \hat{x}_{2f} , \\ \hat{u}_2 &= \hat{x}_{1f} , \\ \hat{u}_3 &= p_3 - \hat{x}_{1f}\hat{x}_{2f} . \end{aligned} \quad (32)$$

Then, the new equivalent system is

$$dx = Fx\,dt + G\,dw_t^1 ,$$
$$dy = C_1 x\,dt + D_1(\hat{x})dt + \sigma\,dw_t^2 , \tag{33}$$

where

$$C_1 = \begin{bmatrix} m(1+\hat{u}_1) & \hat{u}_2 \\ 0 & 1 \end{bmatrix} , \quad D_1(\hat{x}) = \begin{bmatrix} m\hat{u}_3 \\ 0 \end{bmatrix} ,$$

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} , \text{ F,G are the same as in (26).}$$

Now, assume the following:

(i) if f denotes any of the functions A, G, C_1, D_1, σ, then

$$P\left(\int_0^T |f|\,dt < \infty\right) = 1 ;$$

(ii) x_0 given y_0 is conditionally Gaussian. Then, from (6) the corresponding conditionally Gaussian filter is

$$d\hat{x}_1 = \hat{x}_2 dt + \frac{1}{\sigma_1^2}(\Gamma_1(1 + m\hat{u}_1) + m\hat{u}_2\Gamma_3)dv_1 + \frac{\Gamma_3}{\sigma_2^2} dv_2 ,$$
$$d\hat{x}_2 = -\alpha\hat{x}_2 dt + [\frac{1}{\sigma_1^2}(\Gamma_3(1 + m\hat{u}_1) + m\hat{u}_2\Gamma_2)]dv_1 + \frac{\Gamma_2}{\sigma_2^2} dv_2 , \tag{34}$$

where,

$$dv_1 = dy_1 - (1 + \hat{u}_1 m x_1 + m\hat{u}_2\hat{x}_2 + m\hat{u}_3)dt ,$$

$$dv_2 = dy_2 - \hat{x}_2 dt ,$$

and m,α are defined as before.

The covariance equations are

$$d\Gamma_1 = \{2\Gamma_3 - [\frac{1}{\sigma_1^2}(\Gamma_1(1+m\hat{u}_2) + \Gamma_3 m\hat{u}_2)^2 + \frac{\Gamma_3^2}{\sigma_2^2}]\}\,dt\ ,$$

$$d\Gamma_2 = \{\alpha^2 - 2\alpha\,\Gamma_2 - [\frac{1}{\sigma_1^2}(\Gamma_3(1+m\hat{u}_1) + \Gamma_2 m\hat{u}_2)^2 + \frac{\Gamma_2^2}{\sigma_2^2}]\}\,dt\ ,$$

$$d\Gamma_3 = \{\Gamma_2 - \alpha\Gamma_3 - [\frac{1}{\sigma_1^2}(\Gamma_1(1+m\hat{u}_1) + \Gamma_3 m\hat{u}_2)(\Gamma_3(1+m\hat{u}_1) + \Gamma_2 m\hat{u}_2)] - \frac{1}{\sigma_2^2}(\Gamma_2\Gamma_3)\}dt\ , \qquad (35)$$

where $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ are as in (32).

Notice that in this case (34) and (35) are the same as the filter equations of the modified-second-order truncated filter defined by Jazwinski [6] because the nonlinearity is of second order, although the approach is quite different.

Extended Kalman Filter

The filter equations are [6]

$$d\hat{x}_1 = \hat{x}_2 dt + \frac{1}{\sigma_1^2}[(1+m\hat{x}_2)p_1 + p_3 m\hat{x}_1]dv_1 + \frac{p_3}{\sigma_2^2}dv_2\ ,$$

$$d\hat{x}_2 = -\alpha\hat{x}_t dt + \frac{1}{\sigma_1^2}[(1+m\hat{x}_2)p_3 + p_2 m\hat{x}_1)]dv_1 + \frac{p_2}{\sigma_2^2}dv_2\ , \qquad (36)$$

where m,α are as defined before, and

$$dv_1 = (dy_1 - (1 + m\hat{x}_2)\hat{x}_1 dt) ,$$

$$dv_2 = (dy_2 - \hat{x}_2 dt) .$$

The covariance equations are

$$dp_1 = \{2p_3 - \frac{1}{\sigma_1^2} ((1 + \hat{x}_2 m)p_1 + \hat{x}_1 m p_3)^2 - \frac{p_3^2}{\sigma_2^2}\} ,$$

$$dp_2 = \{\alpha^2 - 2\alpha p_2 - \frac{1}{\sigma_1^2} [(1 + m\hat{x}_2)p_3 + m\hat{x}_1 p_2]^2 - \frac{p_2^2}{\sigma_2^2}\} dt , \tag{37}$$

$$dp_3 = \{p_2 - \alpha p_3 - [\frac{1}{\sigma_1^2} ((1 + m\hat{x}_2)p_1 + \hat{x}_1 m p_3)((1 + m\hat{x}_2)$$

$$p_3 + m p_2 \hat{x}_1) - \frac{1}{\sigma_2^2} p_2 p_3\} dt .$$

IV. SIMULATION RESULTS

The TNF algorithm and the EKF algorithm for the previous example were simulated by a digital computer. In the simulation, a fourth-order, Runge-Kutta integration algorithm was used for all trajectory filters and differential equations of both the original system and the error-covariance matrices. Throughout all the simulation cases, the Wiener processes w_t, which describe the excitation noises, were generated from pseudo-random Gaussian variables, v_i, N(0,1). The latter was generated by standard (IMSL) library subroutine, and increments of w_t were approximated by $\Delta w \simeq \sqrt{\Delta t}\, v_i$, where Δt is the integration step size.

The performance of the two filters are compared on the basis of:

(1) The "mean-square error" (m.s.e.) of the filter output to x_t such that

$$J_{TNF} = E\left(\int_0^T (x(t) - \hat{x}_{TNF}(t))^2 dt \right) ,$$
$$J_{EKF} = E\left(\int_0^T (x(t) - \hat{x}_{EKF}(t))^2 dt \right) , \qquad (38)$$

and JJ gives the relative (percentage) difference between J_{TNF}, J_{EKF} such that

$$JJ = \frac{J_{EKF} - J_{TNF}}{J_{EKF}} \times 100 . \qquad (39)$$

(2) Root mean square error

$$Q_j(t) = \left[\sum_{i=1}^{N} \frac{(x_j^{(i)}(t_k) - \hat{x}_j^{(i)}(t_k))^2}{N} \right]^{1/2}, \quad j = 1,2$$

$$VQ_j(t) = \left[\sum_{i=1}^{N} \frac{(x_j^{(i)}(t_k) - \hat{x}_j^{(i)}(t_k))^2}{N} \right]^{1/2}, \quad j = 1,2 \qquad (40)$$

where Q_j, $j = 1,2$ are the RMS range (position) errors. VQ_j, $j = 1,2$ are the RMS velocity (range rate) errors, and $(x_j^{(i)}(t_k), \hat{x}_j^{(i)}(t_k))$ are the j^{th} components of the true state and its corresponding TNF, EKF estimates at time t_k on the i^{th} simulation run, in a series of N runs. For completeness, some comments on the filter initialization seem in order here. Under actual operating conditions it is extremely difficult, and indeed rare due to one reason or another, to obtain reliable initial estimates of the state vector and its associated covariance matrix. Consequently, the following set of initial conditions are reasonably chosen. Throughout, the initial range value is 5000 meters, while the initial range rate value is assumed constant and chosen from the following set (50 m/sec, 500 m/sec, 1000 m/sec). The initial condition of the estimates are calculated according to the following equation:

$$\hat{x}_i(0) = x_i(0) + \sqrt{p_i(0)}\,\eta_i, \quad i = 1,2 \qquad (41)$$

where η_i is a random variable. The initial covariance matrix is

$$P(0) = \begin{bmatrix} p_1(0) & p_3(0) \\ p_3(0) & p_2(0) \end{bmatrix} = \begin{bmatrix} 10^6 & 10^2 \\ 10^2 & 10^4 \end{bmatrix},$$

where the diagonal elements of p(0) are chosen relatively large so that the filter will "forget" the initial values as more data arrived, and to ensure the randomness of the initial estimates. In all cases, a system noise of 1% variance of the initial state values is used, and different levels of measurement noise standard deviation (from 2-20%) of the initial range, range rate respectively, are added. For convenience, the time interval T, for each run is 10 seconds, and the number of runs, N, for each simulation test case is 20. Thus, all results have been ensemble averaged over N = 20 runs.

The effect of increasing the nonlinearity, (i.e., increases in a), of the system on the rms error levels $Q_j(t)$, $VQ_j(t)$, $j = 1,2$, are demonstrated in Figures 4 and 5 as compared to Figures 6 and 7, respectively. Accordingly, the TNF performance improved substantially, and the rms-error levels increased considerably as compared to the rms-error levels of EKF. These comparisons are summarized by Table 1.

Comparison of Figures 8 and 9 with Figures 6 and 7, respectively, and Table 2 indicates that the EKF gains in accuracy relative to the TNF as the observations become more noisy (i.e., increases the range measurement noise standard deviation, σ_1 to 20%). This is due to the fact that the nonlinearity, (here in the range measurement), is masked by the large measurement noise.

From the tables and figures mentioned above, it is seen that in many cases the TNF shows significant improvement in filter accuracy as compared to the EKF. For certain applications, the complexity of the proposed algorithm (TNF) over the EKF would

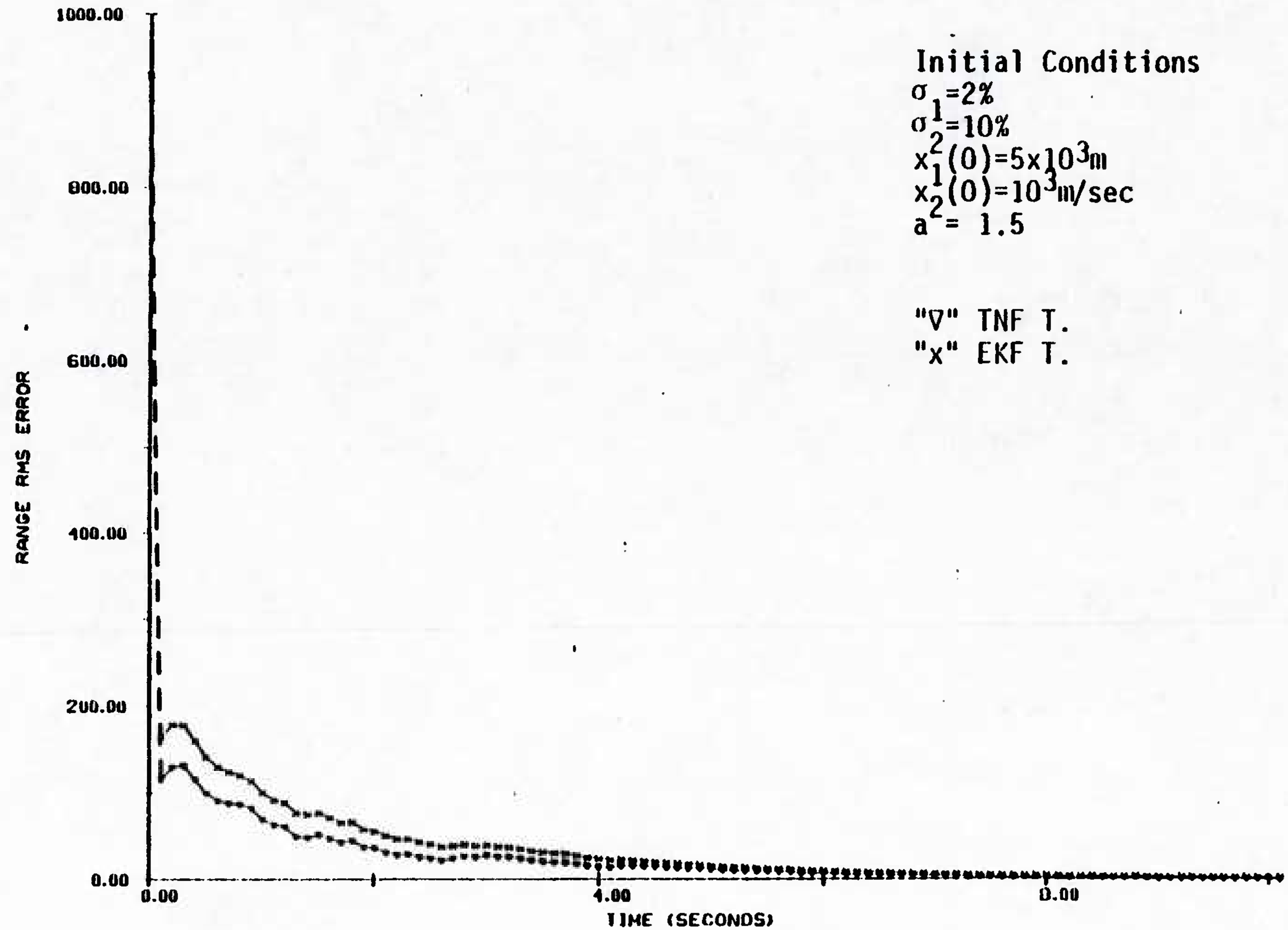


Figure 4. Range RMS Errors by TNF, EKF.

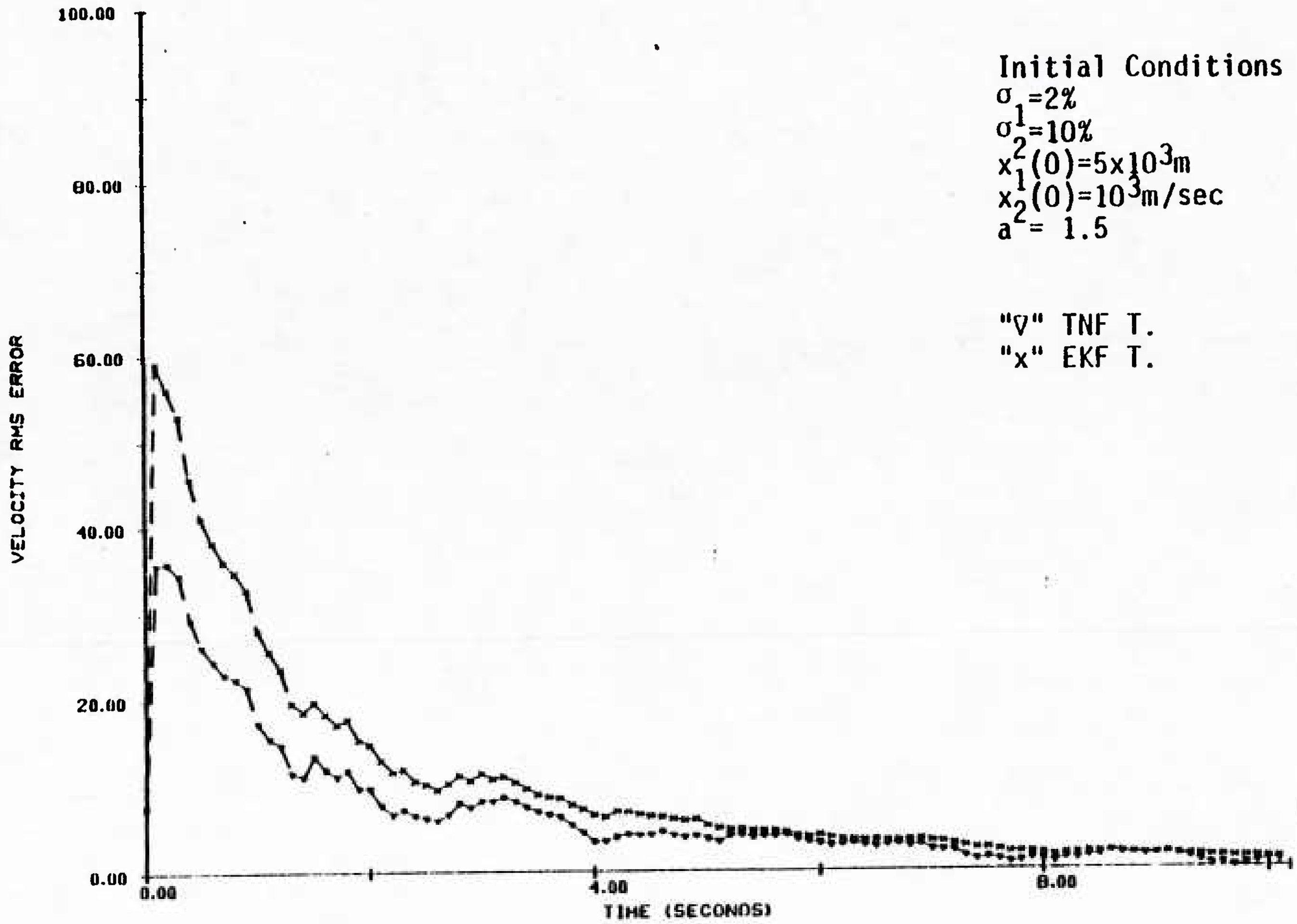


Figure 5. Velocity RMS Errors by TNF, EKF.

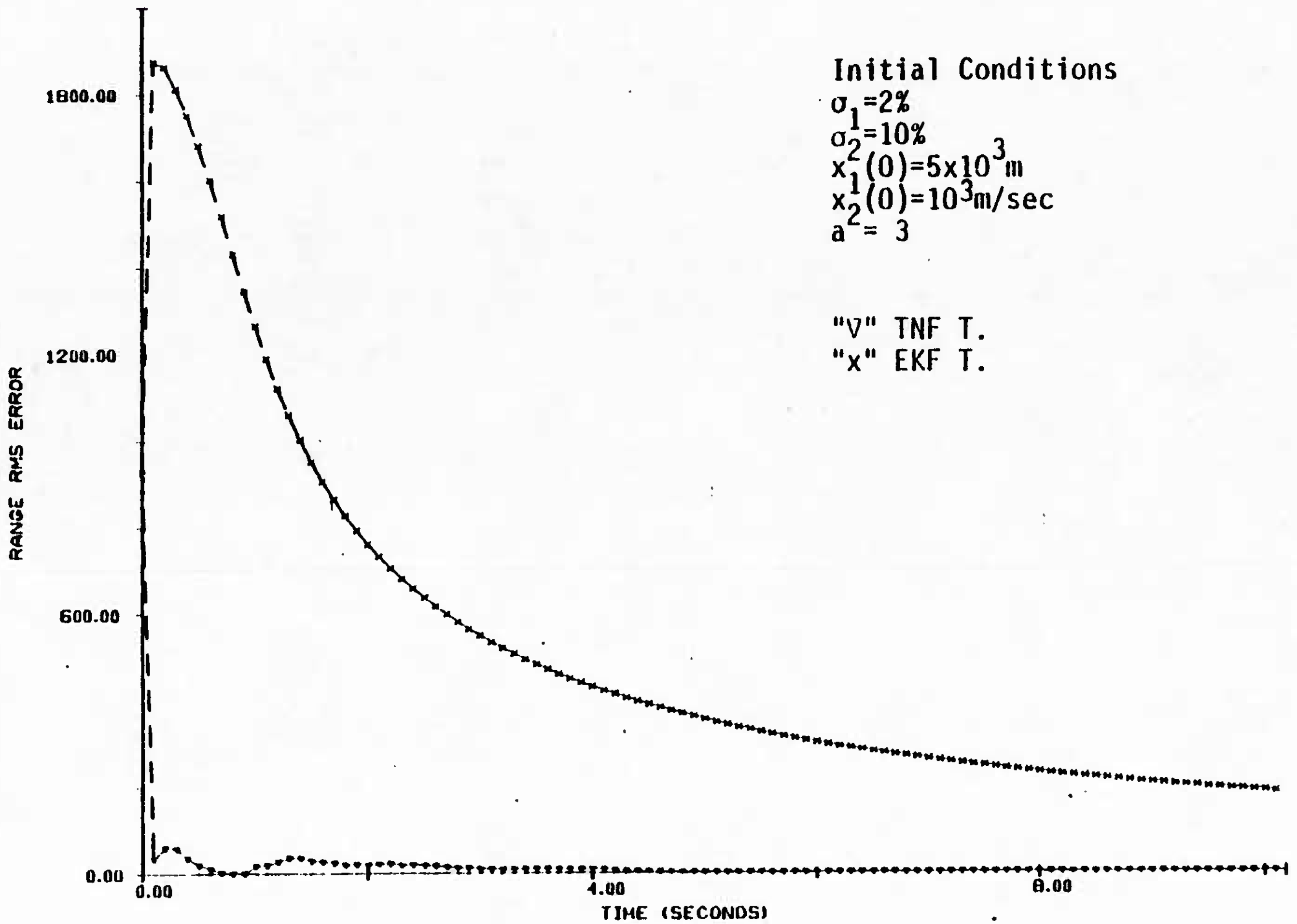


Figure 6. Range RMS Errors by TNF, EKF.

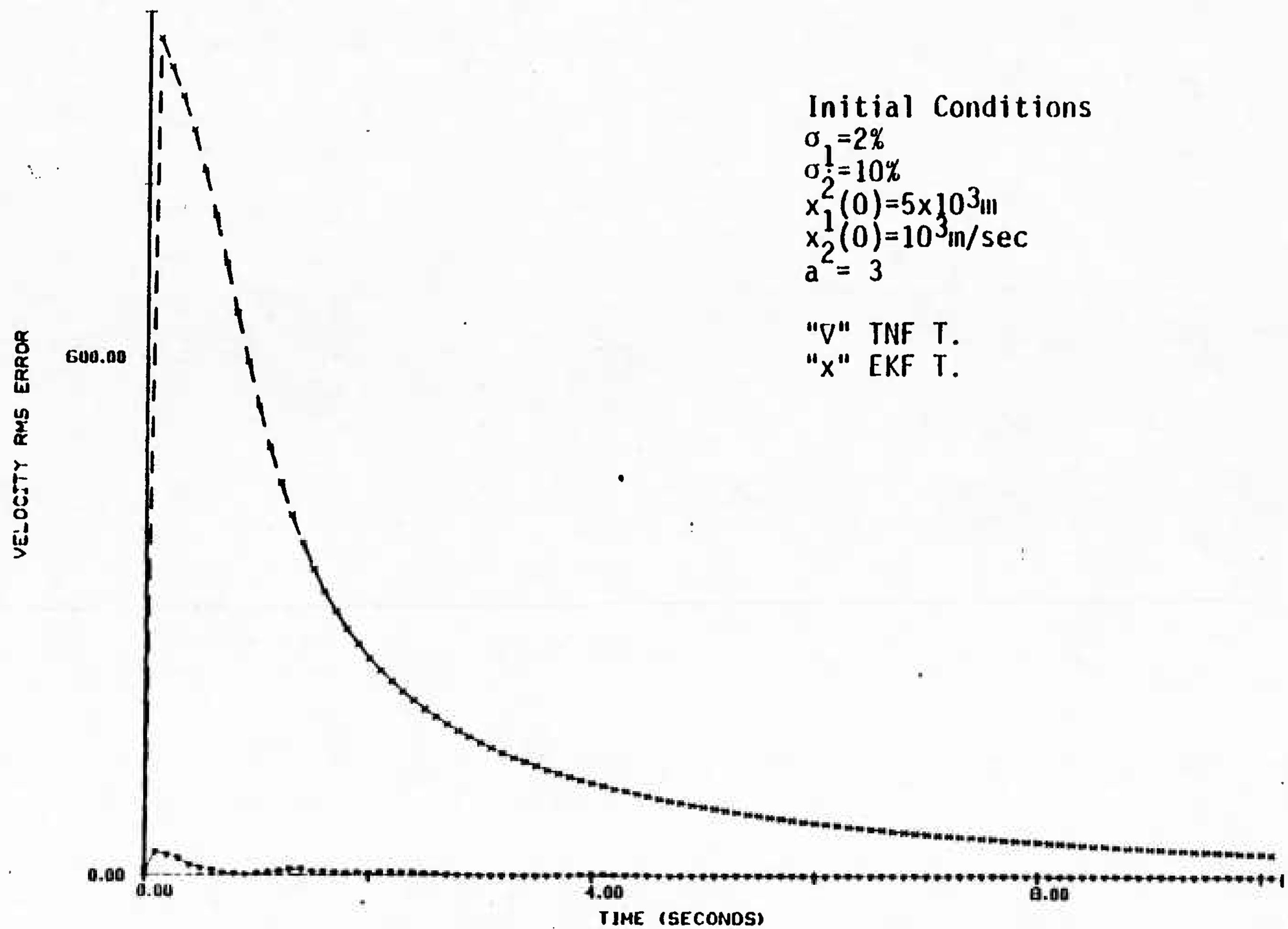


Figure 7. Velocity RMS Errors by TNF, EKF.

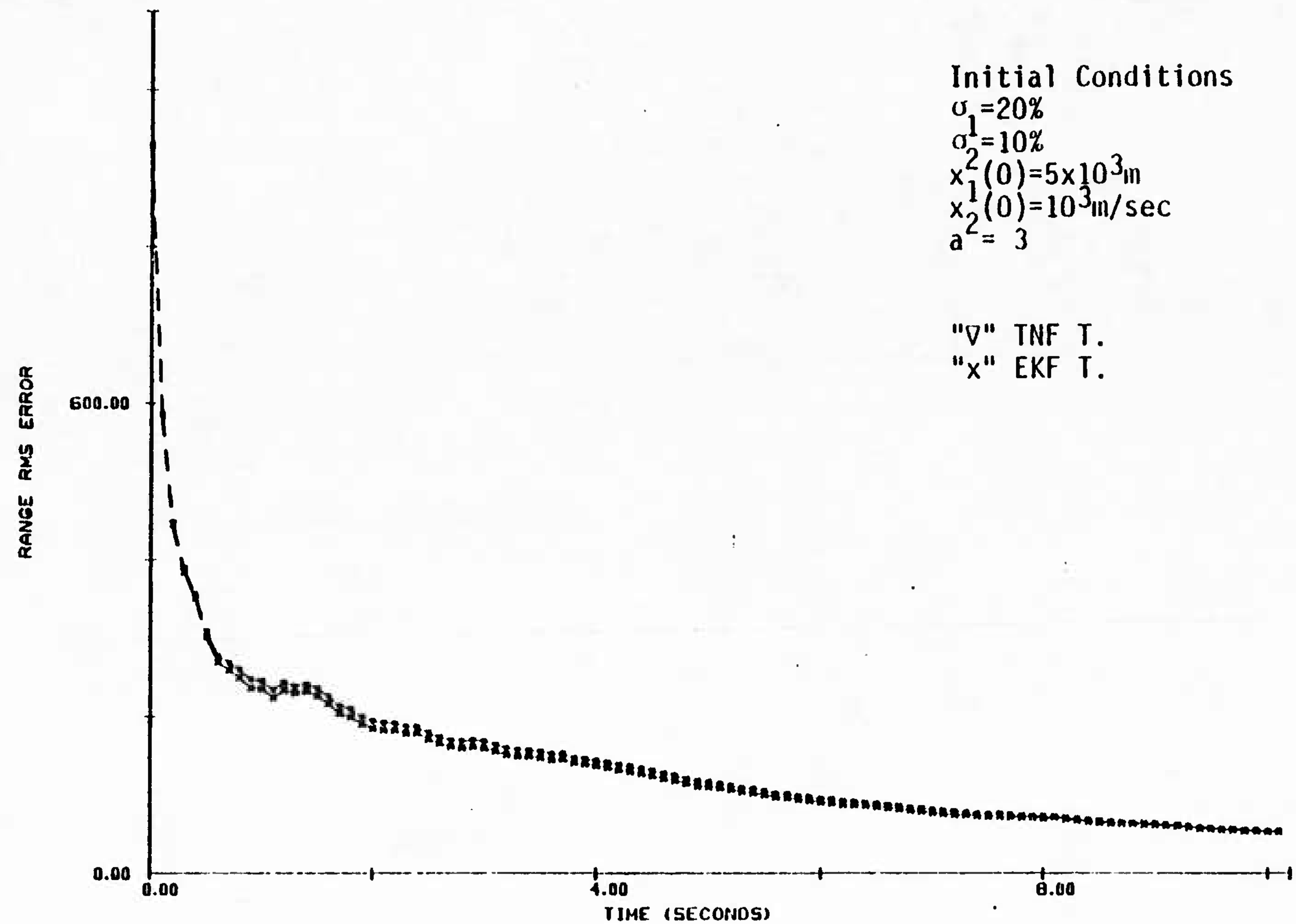


Figure 8. Range RMS Errors by TNF, EKF.

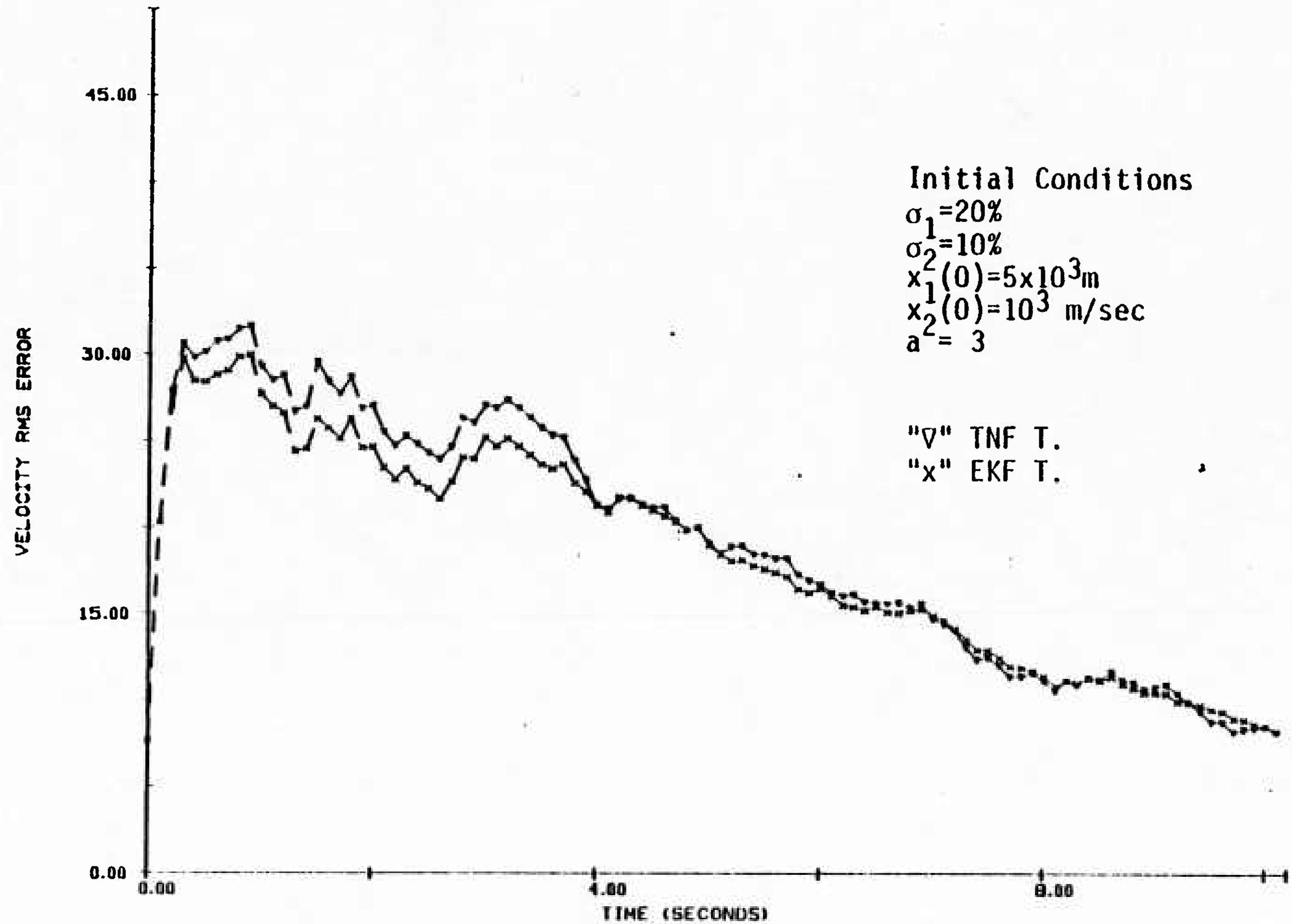


Figure 9. Velocity RMS Errors by TNF, EKF.

Table 1. Synopsis of the Percentage Accuracy of TNF Over EKF.

a	JJ_1%	JJ_2%
1	10.91	35.57
1.5	32.37	57.14
2	78.60	90.03
3	91.42	98.85

(σ_1 = 2%, σ_2 = 10%, $x_1(0) = 5x10^3$m, $x_2(0) = 10^3$ m/sec)

Table 2. The Effect of Measurement Errors on the Percentage Accuracy of the TNF Over the EKF.

a	3	3	3
σ_1	2%	10%	20%
σ_2	10%	10%	10%
JJ_1%	91.42	11.0	- 7.23
JJ_2%	98.03	17.47	-16.13

($x_2(0) = 10^3$ m/sec, $x_1(0) = 5x10^3$m) be justified by the significant improvement in the filter accuracy. That the EKF performs slightly better in the higher observation noise case is due partly at least to the suboptimal approximation used.

V. CONCLUSION

A new global-filtering approximation for a certain class of nonlinear systems is presented. An important practical feature of the proposed method is the method's independence of the model smoothness assumption which is crucial to traditional techniques. Furthermore, a major and equally important byproduct is the generation of a "close" (in m.s.e.) bilinear model approximation of the original nonlinear system. The assumption that x_0 given y_0 is conditionally Gaussian may be satisfied under somewhat realistic operating conditions, and, of course, it is more general than the traditional Gaussian assumption of both x_t and y_t.

The digital computer simulation demonstrates the substantial filtering accuracy improvement of the TNF over the popular EKF in most cases.

In terms of computation time the TNF takes roughly 10% longer than the EKF.

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